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Reconstruction of surfaces of revolution with partial sampling☆

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Abstract

Current surface reconstruction algorithms focus on complex surfaces or irregular surfaces. However, surfaces of revolution are very important in industrial applications and reconstruction of rotating surfaces are often required. To reconstruct a surface of revolution is to determine its axis and generatrix. The main obstacle is derived from, different from surfaces used in current surface reconstruction, the data of a surface of revolution are typically partially sampled. It makes the reconstruction of rotating surfaces a real interesting problem. In this paper, we present an algorithm to compute the axis and generatrix of a surface of revolution in such partially sampled case.

MSC: 65D17; 65D10; 68U07

Keywords: Computational geometry; Surface reconstruction; Surfaces of revolution

1. Introduction

Many applications in CAD, computer graphics and scientific computing involve approximating a surface from its samples. Most of the algorithms proposed in the literature for surface reconstruction focus on complex surfaces or irregular surfaces [1,2,4,7,8,10,11]. However, surfaces of revolution are very important in industrial applications and reconstruction of surfaces of revolution are often required [3].

To reconstruct a surface of revolution is to determine its axis and generatrix [9]. The main obstacle is derived from that the data of a practical surface of revolution are usually partially sampled,

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i.e., sampled merely from a part of the surface of revolution, not from the whole. It makes the reconstruction of rotating surfaces a real interesting problem. In this paper, we present an algorithm to compute the axis and generatrix of a surface of revolution in such partially sampled case.

The paper is organized as follows. In Section 2 we sketch the outline of the algorithm to reconstruct surfaces of revolution. Section 3 is devoted to the curvature argument used in the proposed reconstruction scheme. In Section 4, we give numerical examples to show how our algorithm applies.

2. Outline of the algorithm

Let S be a set of points distributed in $\mathbb{R}^3$:

$$S = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}.$$

We put four assumptions on S.

Assumption A1. *S is δ-noised sampled on a connected surface Σ, where the term "δ-noised" means that for every $\mathbf{x}_i \in S$ there is some $\hat{\mathbf{x}_i} \in \Sigma$ such that $\|\mathbf{x} - \mathbf{x}_i\| < \delta$ for a fixed positive number δ.*

Assumption A2. *S is ε-densely sampled on a surface Σ, where the term "ε-densely" means that for any $\mathbf{x} \in \Sigma$ there is at least one $\mathbf{x}_i \in S$ such that $\|\mathbf{x} - \mathbf{x}_i\| < \varepsilon$ for a fixed positive number ε.*

Assumption A3. *Σ is a dominant part of a surface of revolution $\tilde{\Sigma}$, where the term "dominant part" means that $\tilde{\Sigma}$ can be generated by rotating Σ about the symmetric axis of $\tilde{\Sigma}$.*

Assumption A4. *The parameter ε has been chosen sufficiently small to distinguish the topological type of the original surface of revolution $\tilde{\Sigma}$. In addition, $\delta < \varepsilon$.*

Our target is to compute the symmetric axis and the generatrix of $\tilde{\Sigma}$ via the data set S. Our algorithm can be sketched as follows.

Step 1. (*Partial reconstruction*): Reconstruct a piecewise linear surface Σ' by interpolating S. We apply a Delauney-based method proposed by Dey et al. [5,6]. Clearly Σ' approximates Σ.

Step 2. (*Estimation of the axis*) Select a set of unit vectors $\mathbf{N}_i$ that are uniformly distributed on the unit sphere centered at the origin. For each of the vector $\mathbf{N}_i$, construct a family of parallel planes $\Pi(i,j)$ with the vector $\mathbf{N}_i$ as their common normal direction. Compute the intersections of all $\Pi(i,j)$ and Σ', which consist of a series of piecewise linear curves.

For each i, estimate the average derivative of the curvatures along the intersection curves of $\Pi(i,j)$ and Σ' over all j, and then determine the index i which minimizes the average derivative.

Since Σ is a part of surface of revolution, the minimum average derivative of the curvatures should be close to zero and the intersection curves with respect to it should be approximately circular. For each of the curves compute the center of the relative circle. The symmetric axis of $\tilde{\Sigma}$ is now estimated to be the line L approximately passing through all the centers.

Step 3. (*Estimation of the generatrix*) Select a plane Π with the estimated axis L lying on it. For each sampled point $\mathbf{x}_k (k = 1, \dots, n)$ in S, determine the point $\mathbf{p}_k$ on Π such that $\mathbf{p}_k$ lies on the plane with normal parallel to L and passing through $\mathbf{x}_k$ and

$$d(\mathbf{x}_k, L) = d(\mathbf{p}_k, L),$$

where $d(\mathbf{p}, L)$ denotes the distance between a point $\mathbf{p}$ and the axis L. The generatrix of $\tilde{\Sigma}$ is now estimated to be the planar curve that is reconstructed from the data points $\mathbf{p}_k, k = 1, \dots, n$.

3. The estimation of the symmetric axis

In order to get the symmetric axis of the partially sampled surface of revolution $\tilde{\Sigma}$, we first reconstruct a piecewise linear surface Σ' interpolating the sampled point S. There are many approaches to do this. We choose the method proposed by Dey et al. [5,6], which works well for our goal.

Based on the partial reconstruction surface Σ' of the original surface of revolution $\tilde{\Sigma}$, we can perform the key step of the algorithm to reconstruct the whole surface of revolution, to get the axis. This is derived from the observation that a plane perpendicular to the axis of a surface of revolution intersects the surface on a circle, whose curvature is constant.

3.1. *A curvature argument of discrete planar curves*

Let Γ be a connected piecewise linear curve composing of a set of segments joining two points P_{i-1} and $P_i (i = 1, \dots, m)$, where $P_0, P_1, \dots, P_m$ are coplanar and $m > 2$. There are many ways to estimate the average derivative of the curvatures along Γ. For a sample of them, one may interpolate $P_0, P_1, \dots, P_m$ with a cubic B-spline curve, and then compute the quotient of the integral of the derivative of the curvature of the interpolation curve divided by its length. However, we prefer rendering a local approximate algorithm which can be performed more effectively as follows.

Step 1: The normal vector $\mathbf{N}_i$ of Γ at $P_i, i = 1, \dots, m-1$, is defined as

$$\mathbf{N}_i = \frac{\mathbf{N}'_i}{|\mathbf{N}'_i|}, \quad \mathbf{N}'_i = \frac{\mathbf{n}_i}{|\overline{P_{i-1}P_i}|^2} + \frac{\mathbf{n}_{i+1}}{|\overline{P_iP_{i+1}}|^2},$$

where $\mathbf{n}_i = \mathbf{n} \times \overline{P_{i-1}P_i}$ and $\mathbf{n}$ is the normal of the plane that $P_0, P_1, \dots, P_m$ locate.

Step 2: The curvature of Γ at P_i, $i = 1, \dots, m-2$, is defined as

$$\kappa_i = \frac{\mathbf{N}_{i+1} - \mathbf{N}_i}{|\overline{P_iP_{i+1}}|}.$$

Step 3: The average derivative of the curvature of Γ is defined as

$$\bar{\kappa}' = \frac{\sum_{i=1}^{m-3} |\kappa_{i+1} - \kappa_i|}{\sum_{i=1}^{m-3} |\overline{P_iP_{i+1}}|}.$$

For a piecewise linear curve Γ consisting of more than one connected components $\Gamma_1, \dots, \Gamma_m$, we define the average derivative of the curvature of Γ as the weighted sum

$$\bar{\kappa}' = \sum_{j=1}^{m} \frac{\text{length}(\Gamma_j)}{\text{length}(\Gamma)} \bar{\kappa}'(\Gamma_j),$$

where $\bar{\kappa}'(\Gamma_j)$ denotes the average derivative of the curvature of Γ_j.

3.2. Determining the symmetric axis

To get the axis, we first seek the tangent direction of the axis. We present the algorithm as follows.

Step 1: Select two natural numbers N and M and denote by $\mathbf{N}(\theta,\phi)$ the normal vector $(\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$. For each normal vector

$$\mathbf{N}(k\pi/M, h\pi/M-\pi/2),\quad k=0,1,\ldots,2M-1,\quad h=0,1,\ldots,M,$$

compute the projection $\mathrm{proj}(\Sigma';k,h)$ of Σ' on the line

$$t\mathbf{N}(k\pi/M, h\pi/M),\quad -\infty<t<\infty.$$

Step 2: For each pair (k,h), $k=0,1,\ldots,2M-1$ and $h=0,1,\ldots,M$, $\mathrm{proj}(\Sigma';k,h)$ is a segment. We take a uniform partition of $\mathrm{proj}(\Sigma';k,h)$ with knots $E_0,E_1,\ldots,E_N$, where E_0 and E_N denote the two extremities of $\mathrm{proj}(\Sigma';k,h)$, respectively. We construct $N+1$ parallel planes $\Pi(k,h;i)$ passing through E_i for $i=0,1,\ldots,N$, respectively, and with the vector $\mathbf{N}(\theta,\phi)$ as their common normal direction.

Step 3: For each pair (k,h), $k=0,1,\ldots,2M-1; h=0,1,\ldots,M$ and each i, $i=0,1,\ldots,N$, we compute the intersections of all $\Pi(k,h;i)$ and Σ', denoted by $\Gamma(k,h;i)$, which are piecewise linear curves consisting of one or more connected components. We compute the average derivatives $\bar{\kappa}'(\Gamma(k,h;i))$ of the curvature along such curves $\Gamma(k,h;i)$ and then compute the average derivative

$$\bar{\kappa}'(k,h):=\frac{1}{N+1}\sum_{i=0}^{N}\bar{\kappa}'(\Gamma(k,h;i)).$$

Then we determine the pair (k_0,h_0) such that

$$\bar{\kappa}'(k_0,h_0)=\min_{k,h}\bar{\kappa}'(k,h).$$

Step 4: The normal vector $\mathbf{N}(k_0\pi/M, h_0\pi/M-\pi/2)$ is an approximation of the tangent direction of the axis. To improve the precision, for each normal vector

$$\mathbf{N}(\theta_{k_0,k},\phi_{h_0,h}),\quad k,h=0,1,\ldots,2M-1,$$

where $\theta_{k_0,k}=(k_0-1)\pi/M+k\pi/M^2$, $\phi_{h_0,h}=(h_0-1)\pi/M-\pi/2+h\pi/M^2$, perform Step 1 to Step 3 as above to determine the pair (k_1,h_1) such that

$$\bar{\kappa}'(k_0,h_0;k_1,h_1)=\min_{k,h}\bar{\kappa}'(k_0,h_0;k,h),$$

where $\bar{\kappa}'(k_0,h_0;k,h)$ denotes the average derivative of the curvature of the intersection curves of Σ' and the $N+1$ parallel planes with the common normal vector $\mathbf{N}(\theta_{k_0,k},\phi_{h_0,h})$. We perform this process recursively until either the resulting average derivative cannot be improved or it is less than a threshold preassigned and then obtain a unit vector $\mathbf{N}(\bar{\theta},\bar{\phi})$ which is in the tangent direction of the axis.

Step 5: To determine a point at the axis, recall that at the previous Step $N+1$ parallel planes with $\mathbf{N}(\bar{\theta},\bar{\phi})$ as their common normal have been chosen. Select one of the parallel planes and denote by $\bar{\Pi}$. Divide each connected component of the intersection curves of Σ' and the planes into two arcs with equal length, and then draw the perpendicular bisectors of the chords of the arcs. We project all the bisectors onto $\bar{\Pi}$ and get a set of intersection points. Apply the method of least squares(MLS) to determine the point P which minimizes the sum of the distances squared from P to all the intersection points. We set the line passing through P and with tangent vector $\mathbf{N}(\bar{\theta}, \bar{\phi})$ as the axis of the surface of revolution $\tilde{\Sigma}$.

4. The estimation of the generatrix

Having the axis determined, to estimate the generatrix becomes a comparatively easy task. We present the algorithm as follows:

Step 1: Set up an XYZ-coordinate system with the axis as the Z-axis. For each point $\mathbf{x}_i \in S(i = 1, \ldots, n)$, set $\mathbf{p}_i$ in the XZ-plane such that

$$z(\mathbf{p}_i) = z(\mathbf{x}_i)$$

and the distance from $\mathbf{x}_i$ to Z-axis is equal to the X-coordinate of $\mathbf{p}_i$, where $z(\mathbf{p}_i)$ and $z(\mathbf{x}_i)$ denote the Z-coordinates of $\mathbf{p}_i$ and $\mathbf{x}_i$, respectively.

Step 2: Now we have to reconstruct a planar curve on the XZ-plane with $\mathbf{p}_1, \ldots, \mathbf{p}_n$ as its sampled points. We first construct the Delauney triangles over the scattered points $\mathbf{p}_1, \ldots, \mathbf{p}_n$.

Step 3: We delete all the edges with length greater than $\varepsilon + \delta$ mentioned in the assumptions in Section 2. Naturally, some of the Delauney triangles also disappeared.

Step 4: We delete all the edges that are not edges of any triangles left, and then delete all the edges that are the common edges of two triangles left. The union of the Delauney triangles left is a connected domain in the XZ-plane. The boundary of the domain consists of one or more loops. Denote by Υ the outer loop, where the term "outer loop" means that each of the other loops, if there exist, is contained in the interior of it. Denote by G the interior of Υ. G is a simply connected domain.

Step 5: Select arbitrarily a point $\mathbf{c}_0$ in G and then denote by D_0 the disc centered at $\mathbf{c}_0$ with radius ε. $G \setminus D_0$ consist of at most two connected components under Assumption 4 in Section 2. Let G_1 be one of the components. Select a point $\mathbf{c}_1$ in G_1 such that $\|\mathbf{c}_1 - \mathbf{c}_0\| < 2\varepsilon$ and then denote by D_1 the disc centered at $\mathbf{c}_1$ with radius ε. Recursively, while $\mathbf{c}_j$ has been chosen, select $\mathbf{c}_{j+1}$ in $G_{j+1} := G_j \setminus D_j$ such that $\|\mathbf{c}_{j+1} - \mathbf{c}_j\| < 2\varepsilon$ and then denote by D_{j+1} the disc centered at $\mathbf{c}_{j+1}$ with radius ε until the $G_{j+1} = \emptyset$. Then we obtain a series of points $\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{m_1}$ for some natural number m_1.

Step 6: Perform a similar process to the other component G_2 of $G \setminus D_0$, if there exists, and get a series of points $\mathbf{c}_{-1}, \mathbf{c}_{-2}, \ldots, \mathbf{c}_{-m_2}$ with $\|\mathbf{c}_{-j-1} - \mathbf{c}_{-j}\| < 2\varepsilon$ for $j = 0, 1, \ldots, m_2 - 1$, where m_2 is a natural number.

Step 7: For each disc D_j, compute the centroid $\mathbf{q}_j$ of the point set $S_j = S \cap D_j$, where $j = -m_2, \ldots, -1, 0, 1, \ldots, m_1$.

Step 8: To complete the reconstruction of $\tilde{\Sigma}'$, we interpolate $\mathbf{q}_{-m_2}, \mathbf{q}_{-m_2+1}, \ldots, \mathbf{q}_{m_1}$ with a cubic B-spline curve (cf. [12, Chapter 9]). This B-spline curve is regarded as the generatrix in the sense of approximation.

5. Numerical examples

We wrote a C++ program to verify the feasibility and effectiveness of the provided algorithm and experimented with several models on a PC with 733 MHz Pentium III processor and 128 MB RAM. Partially due to the fact that the sizes of the models we used are quite small, the computing time is at the level of milliseconds. The output axes and generatrices are almost perfectly accurate. Figs. 1 and 2 show the main steps of the provided algorithm.

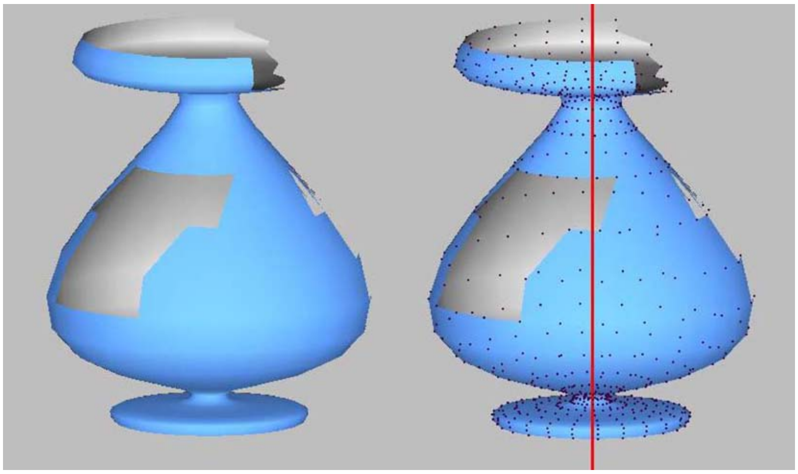

Fig. 1. Left: A part of a surface of revolution. Right: Determine the axis from the sampled data set.

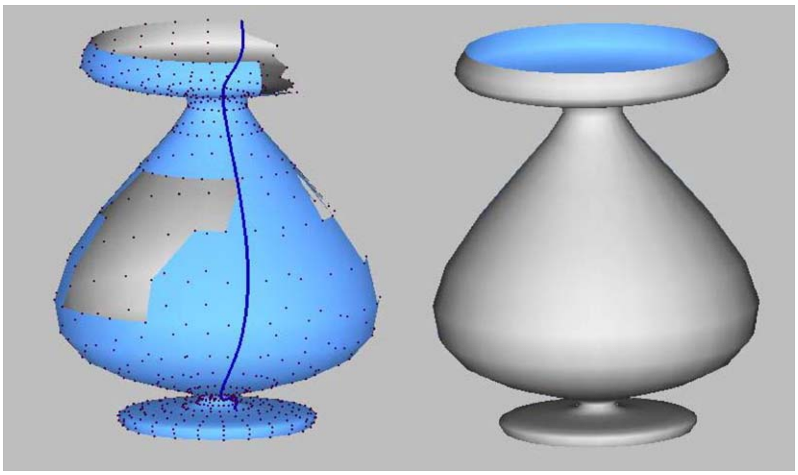

Fig. 2. Left: Determine the generatrix from the sampled data set. Right: The reconstructed surface of revolution.

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